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AN APPLICATION OF MODULAR EQUATIONS IN ANALYSIS SITUS.*

BY OSWALD VEBLEN.

1. By a map we mean a set of α_2 simply connected regions (countries) covering the surface of a sphere and bounded by α_1 simple arcs (edges) joining α_0 distinct points (vertices). No two regions have a point in common, no two arcs intersect.

For all known maps it is possible to assign to each region one of four colors in such a way that any two regions having an edge in common are differently colored. Whether or not this is true for all maps is still unknown in spite of the investigations of a considerable number of mathematicians.† It is therefore perhaps not without interest to show how the problem can be stated in terms of linear equations in a finite field.

These equations turn out to be of service in describing the elementary properties of the map. In particular they supply us with an easy proof of Euler's formula. We shall first outline the discussion of these equations as they arise in the field of integers reduced modulo two and then show how they connect with the four-color problem when the field is extended to include certain Galois imaginaries.

2. A map can be fully described by means of two matrices.‡ To do this, the vertices are numbered in an arbitrary way from 1 to α_0, the edges from 1 to α_1, and the countries from 1 to α_2. In the first matrix the rows correspond to the vertices and the columns to the edges. A " 1 " appears as the element of the ith row and jth column if the ith vertex is on the jth edge; and a " 0 " appears as the element of the ith row and jth column if the ith vertex is not on the jth edge. We shall denote this matrix by A; it has α_0 rows and α_1 columns. In the second matrix the rows correspond to the edges and the columns to the countries. The element of the ith

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* Read before the American Mathematical Society, April 27, 1912.

† A fairly complete set of references is to be found in the article on "Analysis Situs" in the Encyklopädie der mathematischen Wissenschaften, III, AB 3, p. 177. The most recent paper on the subject, "On the Reducibility of Maps" by G. D. Birkhoff will appear in the American Journal of Mathematics, Vol. 35 (1913).

‡ These matrices are identical on interchanging rows and columns with those employed by Poincaré, Proceedings of the London Mathematical Society, vol. 32, 1900, p. 277, if the + and − signs used by the latter are omitted. The + and − signs are clearly not essential to the determination of a manifold.

row and jth column is 1 if the ith edge is on the jth region and "0" if not. We shall call this matrix B; it has α_1 rows and α_2 columns.

For the map obtained by projecting an inscribed tetrahedron from one of its interior points to the surface of a sphere the matrixes A and B are respectively (cf. Fig. 1):

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

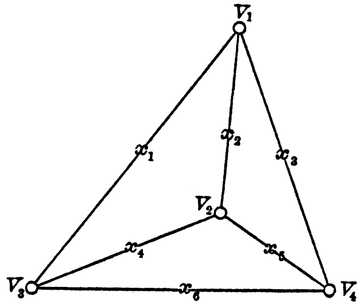

FIG. 1.

3. With these matrices may be associated four sets of linear homogeneous equations. In each case the variables and coefficients are regarded as integers reduced modulo two. In other words, let us add according to the rules $1 + 1 = 0$, $1 + 0 = 1$, $0 + 1 = 1$, $0 + 0 = 0$; and multiply according to the rules $1 \times 1 = 1$, $1 \times 0 = 0$, $0 \times 1 = 1$, $0 \times 0 = 0$. All the formal laws of elementary algebra are satisfied by this field.

In the first set the equations correspond to the rows of the matrix A. There is one variable for each edge of the map and one equation,

$$(1) \qquad x_a + x_b + x_c + \cdots = 0$$

for each vertex, the variables in the equation representing the edges which meet at the corresponding vertex. A solution of this system of equations represents a way of labelling the edges of the map with 0's and 1's so that there shall be an even number of 1's on the edges at each vertex. *The edges labelled with 1's in this manner form a number of closed circuits no two of which have an edge in common.* For let us start with an arbitrary edge labelled 1 and describe a path among the edges labelled 1. Whenever there is an edge by which this path approaches a vertex, since the number of 1-edges at this vertex is even, there is a 1-edge by which the path can go away. Hence the path may be continued till it intersects itself. A portion of the path then forms a closed circuit. If this be removed there are still an even number of 1-edges at each vertex. Another circuit may be removed and so on till all the 1-edges are accounted for.

If $(x_1, x_2, \cdots, x_{\alpha})$ and $(x_1', x_2', \cdots, x_{\alpha}')$ are solutions of the equations (1) it is clear that $(x_1 + x_1', x_2 + x_2', \cdots, x_{\alpha_1} + x_{\alpha_2}')$ is also a solution. The boundary of each of the α_2 countries of the map is represented by a solution in which each edge of the boundary is marked with a 1 and each other edge with a 0. One such solution is supplied by each column of the matrix B. We shall call these the *fundamental solutions*. The solution representing any circuit whatever may be expressed linearly in terms of these α_2 fundamental solutions. In fact, the circuit divides the surface of the sphere into two parts, and the solution representing the circuit is expressible as the sum of the solutions corresponding to the countries in one of these parts.

The sum of the α_2 fundamental solutions is $(0, 0, \cdots, 0)$ because each edge appears on the boundary of two and only two countries. These solutions cannot be subject to any other linear homogeneous relation because the coefficients of such a relation could be only 0 or 1, and hence the relation would merely state that the sum of a certain subset of solutions would be $(0, 0, \cdots, 0)$. This is impossible because any subset of the α_2 countries has at least one country with an edge not on any other country of the subset. Hence *the number of linearly independent solutions of the equations* (1) *is* $\alpha_2 - 1$, *and the total number of solutions is* $2^{\alpha_2 - 1}$.

4. A second set of equations is determined by the columns of the first matrix. In these equations the variables correspond to the vertices of the map and there is one equation

$$v_a + v_b = 0 \tag{2}$$

for each edge, v_a and v_b being the vertices at the ends of the edge. The only possible solutions are such that all the variables are equal. For if v_a is given, v_b must be equal to v_a; if v_c is connected with v_b by an edge, v_c is also equal to v_a, and so on. Since there is a path along the edges joining any vertex to any other it follows by this argument that all the variables are equal to v_a. Hence the only solutions of the equations are $(0, 0, \cdots, 0)$ and $(1, 1, \cdots, 1)$. There are α_0 variables. Hence the number of the α_1 equations which are linearly independent must be $\alpha_0 - 1$. *Hence the rank of the matrix A is $\alpha_0 - 1$.*

5. Let us return a moment to the first set of equations. Here there

were α_0 equations among α_1 variables and there were $\alpha_2 - 1$ solutions. The rank of the matrix A has just been seen to be $\alpha_0 - 1$ so that the number of linearly independent equations is $\alpha_0 - 1$. Hence

$$\alpha_1 - (\alpha_0 - 1) = \alpha_2 - 1,$$

or

$$\alpha_0 - \alpha_1 + \alpha_2 = 2,$$

which is the well-known Euler's formula.

6. The third set of equations may be read from the rows of the matrix B. The variables correspond to the countries and for each edge there is an equation of the form

$$y_a + y_b = 0, \tag{3}$$

where y_a and y_b correspond to the countries meeting in the edge in question. These equations are entirely analogous to the equations (2). In fact if a point (the capital of the country, as Mr. Bennett suggests calling it) be introduced in each region and the points in abutting regions be joined by non intersecting arcs, there is obtained a map (of α_2 points, α_1 edges and α_0 regions) dual to the first map and interchanging the rôles of the matrices A and B. The only solutions of the equations (3) are, by precisely the argument used for the equations (2), $(0, 0, \cdots, 0)$ and $(1, 1, \cdots, 1)$. Hence the rank of B is $\alpha_2 - 1$.

7. The fourth set of equations correspond to the columns of the matrix B. The variables correspond to the edges of the map and for each country there is an equation of the form

$$e_a + e_b + \cdots = 0, \tag{4}$$

where $e_a, e_b, \cdots$ are the edges of the region. Just as in the case of the equations (1), a solution represents a system of circuits among the regions of the map. A circuit is a set of distinct countries $r_1, r_2, \cdots, r_n$ and distinct edges $e_1, e_2, \cdots, e_n$ such that r_1 and r_2 meet along e_1, r_2 and r_3 meet along $e_2, \cdots, r_n$ and r_1 meet along e_n. A circuit is *simple* if no subset of its edges and regions form a circuit. To distinguish a simple circuit composed of edges and regions from an ordinary circuit which is composed of edges and vertices we may call the former a *cycle*. The set of all edges and regions meeting at a vertex form a cycle which we shall call *fundamental*. The solutions which correspond to fundamental cycles are given by the rows of Matrix A of which $\alpha_0 - 1$ are linearly independent. Since the rank of B is $\alpha_2 - 1$ and the number of variables in the equations (4) is α_1, the number of linearly independent solutions is

$\alpha_1 - \alpha_2 + 1$, which by Euler's formula is $\alpha_0 - 1$. Hence the α_0 fundamental cycles furnish a set of solutions of (4) in terms of which all the solutions are expressible; and these fundamental solutions satisfy one linear homogeneous relation.

8. Turning to the four color problem, let us suppose the field $GF(2)$, consisting of 0 and 1 combined modulo two, to be extended by the Galois imaginaries satisfying the relations $i^2 + i + 1 = 0$. The extended field $GF(2^2)$ has four elements, 0, 1, i, $i + 1$ which we may use to denote the four colors. Two elements α, β of this field are equal if and only if $\alpha + \beta = 0$. *Hence a solution of the four color problem consists in finding a set of values* $(y_1, y_2, \cdots, y_{\alpha_2})$ *which satisfies none of the equations* (3) *corresponding to the rows of the matrix B.*

9. The set of values $(y_1, y_2, \cdots, y_{\alpha_2})$ may be regarded as a point in a finite projective space of $\alpha_2 - 1$ dimensions* provided we exclude the set $(0, 0, 0, \cdots, 0)$ and regard $(ky_1, ky_2, \cdots ky_{\alpha_2})$ as the same as $(y_1, y_2, \cdots, y_{\alpha_2})$. Each of the equations (3) then represents an $(\alpha_2 - 2)$-space. If the variables y_i range only over the $GF(2)$ there will be $2^{\alpha_2} - 1$ points in the $(\alpha_2 - 1)$ space. If the variables range over the $GF(2^2)$ there will be $(4^{\alpha_2} - 1)/3$ such points. The first space is included in the second and the points of the second space not included in the first may be regarded as *imaginary* with respect to the first space.

In general, there can be no real point which satisfies none of the equations (3) for such a point would represent a coloring of the map by two colors, which is impossible whenever an odd number of regions meet at any vertex. Hence, in general, every real point lies on at least one of the $(\alpha_2 - 2)$-spaces.

An imaginary point can be written in the form $(y_1 + iy_1', y_2 + iy_2', \cdots, y_{\alpha_2} + iy_{\alpha_2}')$ where $(y_1, y_2, \cdots, y_{\alpha_2})$ and $(y_1', y_2', \cdots, y_{\alpha_2}')$ are real. Hence every imaginary point is on a real line. No imaginary point can be on two real lines because two such lines if they intersect at all have a real point in common. But if $(y_1 + iy_1', y_2 + iy_2', \cdots, y_{\alpha_2} + iy_{\alpha_2}')$ satisfies one of the equations (3), so must $(y_1, y_2, \cdots, y_{\alpha_2})$ and $(y_1', y_2', \cdots, y_{\alpha_2}')$, and conversely. Hence a solution of the four color problem is given *by each real line which does not lie on any of the* $(\alpha_2 - 2)$ *spaces which are represented by equations* (3).

10. If a point of the real $(\alpha_2 - 1)$-space does not satisfy any of the equations (3) corresponding to the edges in a cycle (cf. §7) the countries in the cycle must be assigned alternately the values 0 and 1. This is impossible in a cycle containing an odd number of regions. Hence *every*

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* Cf. Veblen and Bussey, Transactions of the American Mathematical Society, vol. 7 (1906), p. 240.

real point $(y_1, y_2, \cdots, y_{\alpha_2})$ *lies in at least one* $(\alpha_2 - 2)$*-space corresponding to an edge of each odd cycle in the map.*

If the equations corresponding to the edges in a cycle be added, each variable enters twice in the sum. Hence the sum vanishes. In other words, the equations corresponding to the edges in the cycle satisfy a linear relation. If any subset of these equations be added it is clear from the definition of a cycle that there is at least one variable which enters only once in the sum, and hence the sum does not vanish. Hence the equations corresponding to the edges in a cycle satisfy only one linear relation. A set of n $(\alpha_2 - 2)$-spaces in an $(\alpha_2 - 1)$-space would in general meet in an $(\alpha_2 - n - 1)$-space. But the $(\alpha_2 - 2)$ spaces corresponding to the edges in a cycle satisfy one linear relation and therefore meet in an $(\alpha_2 - n)$-space. Let us denote by $S^c_{\alpha_2 - n}$ the $(\alpha_2 - n)$-space thus determined by a cycle C_n of n regions.

Any point of $S^c_{\alpha_2 - n}$ must satisfy all the equations corresponding to edges of C_n. Hence, if n is odd, a line joining any point of $S^c_{\alpha_2 - n}$ to any point whatever must lie in at least one of the $(\alpha_2 - 2)$-spaces corresponding to the edges in C_n. Therefore a line which furnishes a solution of the four color problem cannot pass through any point of the $S^c_{\alpha_2 - n}$ corresponding to any odd chain C_n. In order that it be possible to color the map it is necessary that there be at least one point $(y_1, y_2, \cdots, y_{\alpha_2})$ not on any $S^c_{\alpha_2 - n}$ for which n is odd.

This condition is also sufficient. For suppose we have a point $(y_1, y_2, \cdots, y_{\alpha_2})$ not on any $S^c_{\alpha_2 - n}$ corresponding to an odd value of n. The countries are accordingly all labelled 0 or 1, and every cycle consisting entirely of 0's or entirely of 1's contains an even number of countries. The map breaks up into a finite number of connected portions, each of which is entirely composed of 0-countries and entirely bounded by 1-countries or entirely composed of 1-countries and entirely bounded by 0-countries. Consider all the 0-countries of a given connected set. They can be reached from an arbitrary 0-country, r_1, by paths which do not pass through vertices and go only through 0-countries. The paths from r_1 to any other country r_2 of this set cross always an odd or always an even number of edges; for if not, on cancelling the common edges of two paths from r_1 to r_2 there would remain at least one odd cycles of 0-countries. Let us color white the country r_1 and all countries of the set connected with it which are reached by crossing an even number of edges and let us color black all countries of the set which are reached by crossing an odd number of edges. No two white countries are adjacent nor are any two black countries. Treat all the connected sets of 0-countries in this fashion. Treat all the connected sets of 1-countries similarly with the colors red and yellow. The result is a solution of the four color problem. Hence the existence of a point $(y_1, y_2, \cdots, y_{\alpha_2})$ not on any $S^c_{\alpha_2-n}$ for which n is odd implies a solution of the four color problem.

The four color problem has now been reduced to the following form. In a finite projective space of $(\alpha_2 - 1)$ dimensions with three points on a line there are a certain number of spaces $S^c_{\alpha_2-n}$ of dimensionality $\alpha_2 - n$, one for each odd cycle C_n. They all have one point in common (§6). *The map can be colored in four colors if and only if there exists a point not on any of these* $S^c_{\alpha_2-n}$*'s. There are as many distinct ways of coloring the map (aside from permutations of the colors) as there are real lines in the* $(\alpha_2 - 1)$*-space which do not meet any* $S^c_{\alpha_2-n}$ *(*n*, odd).*

11. Another set of equations associated with the map problem arises as follows. If y_i and y_j are two variables which appear in the same one of the equations (3), i. e., which correspond to adjacent regions, let us denote $y_i + y_j$ by a new variable x_k. There will be one x_k for each of the equations (3), i. e., for each edge of the map. The condition that none of the equations (3) be satisfied now takes the form

$$x_k \neq 0 \quad (k = 1, 2, \cdots, \alpha_1). \tag{5}$$

The set of all x's corresponding to the edges meeting at a vertex of the map is a sum of pairs of y's in which each y appears twice. Hence if the x's meeting at a vertex be $x_a, x_b, x_c, \cdots$, they must satisfy the equation,

$$x_a + x_b + x_c + \cdots = 0. \tag{6}$$

These equations are evidently the same as (1), §3. In other words, *it is necessary in order to solve the problem to find a solution of the equations* (1) *in which none of the variables vanishes.*

This is also sufficient. For any equation of the form (6) in which x_a, $x_b, \cdots$ are the edges which appear in a cycle is linearly dependent on the equations (1). (It is in fact the sum of the equations (1) corresponding to vertices in one of the two parts into which the surface of the sphere is divided by the cycle.) Hence a solution of the equations (1) is such that the sum of the marks on the edges of any cycle is zero. Suppose this solution labels all the edges of the map with marks different from 0. If now the mark 0 be assigned to an arbitrary region and the other regions be marked according to the rule that the sum of the marks of two adjacent countries shall be equal to the mark of the edge separating them, a unique mark is assigned to each country; otherwise the edges of some cycle would have a sum different from zero. Since the marks of no two adjacent countries are alike, this determines a coloring in four colors.

12. It is well known that the four color problem may be reduced to the problem of coloring a map in which three edges meet at each vertex. In this case the equations (1) discussed in §11 have only three turns each. This form of the equations was found by Mr. A. A. Bennett by a method different from the above. Mr. Bennett however did not regard the variables as marks of a Galois Field.

From the equations (1) in this form can be derived the set of equations modulo three discovered by Heawood.* In any solution of (1) the three values of the variables x_a, x_b, x_c corresponding to the edges which meet at any vertex must be 1, i and i^2 in some order, for this is the only way of satisfying

$$x_a + x_b + x_c = 0$$

by values different from zero. An arbitrary sense on the sphere having been chosen as positive, the three marks on the edges at any vertex follow one another in the positive sense either in the cyclic order 1, i, i^2 or the cyclic order 1, i^2, i. In the first case each mark is obtained from its predecessor by multiplying by i, in the second case by multiplying by i^2. Thus by properly distributing the marks i, and i^2 at the vertices the marks of all edges are determined as soon as the mark on one edge is given. For if the mark α on an edge, e, is given and β is the mark at one of its vertices, v, then the mark on the edge following e in the positive cyclic order at v is $\alpha\beta$ and the mark on the other edge is $\alpha\beta^2$. In order that this process assign a unique mark to each edge, the product of the i's and i^2's multiplied in while describing any closed circuit must be unity.

It is necessary and, in view of the simple connectivity of the sphere, sufficient that this condition be satisfied for the boundary of each country in the map. In view of the identity,

$$i^3 = 1,$$

this means that the sum of the exponents of the i's at the vertices of any country must be divisible by three. Hence if $z_1, z_2, \cdots, z_k$ are the exponents at the vertices of any country they must satisfy the relation,

$$z_1 + z_2 + \cdots + z_k = 0 \qquad \text{(mod. 3)}.$$

The α_2 countries give rise to α_2 equations of this form among α_0 variables representing the vertices of the map. These are Heawood's equations. The matrix of the coefficients of the equations is the matrix analogous to A and B representing the incidence relations of the vertices and countries of the map.

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[*] Quarterly Journal of Pure and Applied Mathematics, vol. 29 (1898), p. 270.

To solve the four color problem it is necessary and sufficient to find a solution of these equations in which none of the variables vanish. The variables may be interpreted as coördinates of points in a finite projective space of α_0-dimensions in which there are four points on every line.